

# VECTOR SPACES AND HETEROGENEOUS RAMSEY ALGEBRAS

ZU YAO TEOH AND WEN CHEAN TEH

**ABSTRACT.** Carlson introduced the notion of a Ramsey space as his generalization to the Ellentuck space. When a Ramsey space is induced by an algebra, Carlson had suggested a study of its purely combinatorial version—Ramsey algebra. This study was first picked up by Teh and basic results for homogeneous algebras have been obtained. In this paper, we introduce the notion of a Ramsey algebra for heterogeneous algebras. Then, we study the Ramsey algebraic properties of vector spaces.

## 1. INTRODUCTION

The notion of a Ramsey space was introduced by Carlson in [2]. The class of Ramsey spaces induced by algebras has been singled out to be studied from a purely combinatorial point of view. An algebra is simply a structure consisting of a family of sets, called the domain of the algebra, and a family of operations on these sets. Hindman's Theorem says that if we distribute all the natural numbers into finitely many pigeonholes, we may extract an infinite subcollection of elements whose (finite) sums can be found in the same pigeonhole. The same theorem can be cast in terms of finite subsets of the natural numbers, namely if we partition the collection of finite subsets of the natural numbers into finitely many pieces, we can find an infinite collection of finite subsets that are pairwise disjoint and for which any finite union of these finite sets are in the same piece of the partition [7]. Considering questions about infinite sets along the same line, Erdős-Rado [4] has showed that, unlike the case for finite sets, not every set of reals<sup>1</sup> has the analogous property possessed by sets associated to the foregoing theorem. Hence, it is natural to ask which among those definable sets—specifically, sets belonging to the  $\sigma$ -algebra generated by the usual topology on the reals—have such analogous property. Galvin-Prikry [5] showed that the Borel sets have such analogous property, whereas, using the method of forcing, Silver [8] showed that the analytic sets have the desired property.

Later on, Ellentuck in [3] showed that the result of Silver could also be proved based on topology alone. In his proof, Ellentuck introduced what is to be called the Ellentuck topology, which was then picked up and generalized by Carlson in his work on Ramsey spaces [2]. The merit of working in this more general framework, as Carlson showed, is that classical results such as the Hales-Jewett

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<sup>1</sup>We are adopting the convention in Set Theory, where the infinite subsets of natural numbers are identified with the real numbers.

Theorem and Hindman's Theorem itself came as corollaries to his results on the Ramsey space of variable words.

If one is to ask whether a given structure is a Ramsey space, the definition requires that one checks some topological properties of the space. However, Carlson's abstract version of Ellentuck's Theorem (cf. [2]) turns such topological question into a combinatorial one. Carlson had, in fact, suggested that one could actually embark on a specifically combinatorial study of those spaces induced by algebras as mentioned earlier; this study was then initiated by Teh [cf. [9] & [10]] and the topic became known as Ramsey algebra. Ramsey algebra can thus be viewed as the combinatorial counterpart of Ramsey space that circumvents all topological notions.

Previous works on Ramsey algebras have included, among others, the study of infinite integral domains [9]. In this paper, we want to extend the study of Ramsey algebras to vector spaces. Works on Ramsey algebra had previously been focused on algebraic structures whose domains are of a unique sort. As for vector spaces, we have the set of vectors as well as the set of scalars; to accommodate this two-sorted structure, we will first be formulating a notion of Ramsey algebra for many-sorted algebras, then we will derive some results pertaining to vector spaces. A latter section is then dedicated to the study of the homogeneous structure over a family of functions induced by the nonzero scalars of a vector space.

## 2. PRELIMINARY

This paper concerns the study of Ramsey algebraic properties of vector spaces. A vector space is a structure  $(\mathbb{V}, \mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}}, +_{\mathbb{V}}, \cdot)$ , where  $(\mathbb{V}, +_{\mathbb{V}})$  forms an abelian group,  $(\mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}})$  a field whose elements are called scalars, and  $\cdot : \mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$  is scalar multiplication such that  $1 \cdot v = v$  for all  $v \in \mathbb{V}$ , and the following axioms of distributivity hold:

- (1)  $(r +_{\mathbb{F}} s) \cdot v = r \cdot v +_{\mathbb{F}} s \cdot v$  for each  $r, s \in \mathbb{F}, v \in \mathbb{V}$ ;
- (2)  $r \cdot (u +_{\mathbb{V}} v) = r \cdot u +_{\mathbb{V}} r \cdot v$  for each  $r \in \mathbb{F}, u, v \in \mathbb{V}$ .

When we say that  $\mathcal{F}$  is a family of operations or functions on the family of sets  $(A_{\xi})_{\xi \in I}$ , we mean that for each  $f \in \mathcal{F}$ , there exist  $\xi, \xi_1, \dots, \xi_n$  such that  $f : A_{\xi_1} \times \dots \times A_{\xi_n} \rightarrow A_{\xi}$ . An *algebra* is an ordered pair  $((A_{\xi})_{\xi \in I}, \mathcal{F})$  consisting of a family  $(A_{\xi})_{\xi \in I}$  of sets and a family  $\mathcal{F}$  of operations on the family  $(A_{\xi})_{\xi \in I}$ . The family of sets pertaining to a given algebra is called the *universe* or *domain* of the algebra and each set in the family a *phylum* (cf. [1]). Throughout this paper, we reserve  $I$  to denote the indexing set in the algebra under discussion.

When  $I$  is a singleton, the algebra is said to be *homogeneous*. Examples of homogeneous algebras are groups, rings, Boolean algebras, etc.. For a homogeneous algebra, the notation  $((A_{\xi})_{\xi \in I}, \mathcal{F})$  will be simplified to  $(A, \mathcal{F})$ , where  $A$  is the only phylum of the domain. A vector space is a good example of a heterogeneous algebra consisting of two phyla; an example with three phyla is an automaton, whose phyla are the sets of inputs, states, and outputs.

The set of natural numbers, inclusive of 0, is denoted by  $\omega$ . If  $f$  is a function,  $\text{Dom}(f)$  denotes the domain of  $f$  and  $\text{Rn}(f)$  denotes the range of  $f$ . We will identify any  $n$ -tuple  $(x_1, \dots, x_n)$  with the finite sequence  $\langle x_1, \dots, x_n \rangle$  and, if  $\bar{x}$

denote the  $n$ -tuple,  $\vec{x}$  will denote the associated sequence, and vice versa. If  $\vec{e}$  is an infinite sequence, the notation  $\vec{e} - n$  denotes the tail  $\langle \vec{e}(n), \vec{e}(n+1), \dots \rangle$ ; in particular,  $\vec{e} - 0 = \vec{e}$  for each infinite sequence  $\vec{e}$ . If  $\sigma$  is a finite sequence, then  $|\sigma|$  denotes the length of the sequence.

The uppercase “oh”  $O$  is reserved for the zero vector in any vector space under discussion. If  $D$  is a set, then  $\text{id}_D$  denotes the identity function on  $D$ .

While it is in general not a requirement for the family of phyla  $(A_\xi)_{\xi \in I}$  in a given heterogeneous algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$  to be pairwise disjoint, we shall always assume that the family *is* pairwise disjoint in the study of heterogeneous Ramsey algebra following the assumption made by Carlson in [2]. In our paper, if  $A_0$  and  $A_1$  are phyla of a vector space, the former always denotes the set of scalars and the latter the set of vectors.

Throughout this paper, we will often tacitly assume that the combination of a function  $f$  and a tuple  $\bar{b}$  are suitable for each other in the sense that if  $f : A_{\xi_1} \times A_{\xi_2} \times \dots \times A_{\xi_n} \rightarrow A_\xi$  and we write  $f(\bar{b})$ , then  $\bar{b} \in A_{\xi_1} \times A_{\xi_2} \times \dots \times A_{\xi_n}$  *a priori*.

### 3. THE BASIC NOTIONS

In this section, we want to define the basic notions involved in the study of Ramsey algebra. We will begin with the notion of an orderly composition due to Carlson.

**Definition 3.1 (Orderly Composition & Orderly Terms).** Let  $\mathcal{F}$  be a family of operations on  $(A_\xi)_{\xi \in I}$ . An  $n$ -ary function  $f$  is called an *orderly composition* of  $\mathcal{F}$  if there exists  $h_1, \dots, h_k, g \in \mathcal{F}$  such that

- (1)  $g$  is a  $k$ -ary function,
- (2)  $h_j$  is an  $n_j$ -ary function for each  $j \in \{1, \dots, k\}$ ,
- (3)  $\sum_{j=1}^k n_j = n$ , and
- (4) if  $\bar{x}_1 = (x_1, \dots, x_{n_1})$  and  $\bar{x}_j = (x_{\sum_{i=1}^{j-1} n_i + 1}, \dots, x_{\sum_{i=1}^j n_i})$  for each  $j = \{2, \dots, k\}$ , then  $f(x_1, \dots, x_n) = g(h_1(\bar{x}_1), \dots, h_k(\bar{x}_k))$ .

The collection  $OT(\mathcal{F})$  of *orderly terms* over  $\mathcal{F}$  is the *smallest* collection of functions containing  $\mathcal{F} \cup \{id_{A_\xi}\}_{\xi \in I}$  and is closed under orderly compositions.

The collection of orderly terms over  $\mathcal{F}$  is in fact the collection of operations on  $(A_\xi)_{\xi \in I}$  which can be generated by an application of finitely many of the following rules:

- (1) for each  $\xi \in I$ , the identity function  $\text{id}_{A_\xi}$  is an orderly term,
- (2) every operation in  $\mathcal{F}$  is an orderly term,
- (3) if  $f$  is an operation on  $(A_\xi)_{\xi \in I}$  given by  $f(\bar{x}_1, \dots, \bar{x}_k) = g(h_1(\bar{x}_1), \dots, h_k(\bar{x}_k))$  for some  $g \in \mathcal{F}$  and some orderly terms  $h_1, \dots, h_k$ , then  $f$  is an orderly term.

**Definition 3.2 (Reduction  $\leq_{\mathcal{F}}$ ).** Let  $((A_\xi)_{\xi \in I}, \mathcal{F})$  be an algebra and let  $\vec{a}, \vec{b}$  be infinite sequences of  $\bigcup_{\xi \in I} A_\xi$ . Then  $\vec{a}$  is said to be a *reduction* of  $\vec{b}$ , written  $\vec{a} \leq_{\mathcal{F}} \vec{b}$ , if for each  $j \in \omega$ , there exist orderly terms  $f_j$  over  $\mathcal{F}$  and subsequences  $\vec{b}_j$  of  $\vec{b}$  such that

- (1)  $\vec{a}(j) = f_j(\vec{b}_j)$  and
- (2)  $\vec{b}_0 * \vec{b}_1 * \dots$  forms an infinite subsequence of  $\vec{b}$ .

The notation  $*$  is the concatenation operation. Observe that if  $\vec{a}(j) = f_j(\vec{b}_j)$  and  $\vec{b}_j = \langle \vec{b}(j_1), \dots, \vec{b}(j_k) \rangle$ , then  $j \leq j_1$ ; this is a fact that will be used repeatedly in the proofs appearing in Section 5. Also observe that if every initial segment of  $\vec{a}$  is a “reduction”<sup>2</sup> from an initial segment of  $\vec{b}$  in the following sense, then  $\vec{a} \leq_{\mathcal{F}} \vec{b}$ : if  $\vec{a} \upharpoonright N$  is an initial segment of  $\vec{a}$ , then for each natural number  $j \leq N$ , there exist an orderly term  $f_j$  and a finite subsequence  $\vec{b}_j$  of  $\vec{b}$  such that

- (†)  $\vec{a}(j) = f_j(\vec{b}_j)$  for each  $j \leq N$  and
- (‡)  $\vec{b}_0 * \dots * \vec{b}_N$  forms a subsequence of  $\vec{b}$ .

(Cf. Lemma 2.4 of [11].) This fact is particularly handy when we are to construct a reduction  $\vec{a}$  from a given sequence  $\vec{b}$ ; see, in particular, the proof of Proposition 5.2.

The relation  $\leq_{\mathcal{F}}$  is a preorder. It is interesting to note that if the identity functions are not included in the base set of orderly terms, then the identity functions are not necessarily orderly terms anymore and the relation  $\leq_{\mathcal{F}}$  need not be reflexive. This is illustrated by the sequence  $\vec{b}$  of  $\mathbb{Z}^+$  consisting of the constant term 1; without the inclusion of the identity functions, we would have  $\vec{b} \not\leq_{\{+\}} \vec{b}$ . As further illustration, observe that orderly terms over  $\{+\}$  are each obtained from an iteration of the binary operation  $+$ ; for instance,  $f(x, y, z) = +(+(x, y), \text{id}_{\mathbb{Z}^+}(z)) = x + y + z$  is an orderly term over  $\{+\}$ . In the absence of the identity function  $\text{id}_{\mathbb{Z}^+}$ , this would have been a nonexample. Based on Definition 3.2, each reduction of a given infinite sequence  $\vec{b}$  consists of terms which are (finite) sums of the terms of  $\vec{b}$  obtained in an orderly fashion. To be more precise, if  $\vec{a} \leq_{\{+\}} \vec{b}$ , then for each  $i \in \omega$ , there exist  $j_m > \dots > j_1 > i_n > \dots > i_1 \geq i$  such that  $\vec{a}(i) = \vec{b}(i_1) + \dots + \vec{b}(i_n)$  and  $\vec{a}(i+1) = \vec{b}(j_1) + \dots + \vec{b}(j_m)$ .

**Definition 3.3 (Finite Reduction).** Suppose  $((A_{\xi})_{\xi \in I}, \mathcal{F})$  is an algebra and  $\vec{b} \in {}^{\omega}(\bigcup_{\xi \in I} A_{\xi})$ , we call  $a \in \bigcup_{\xi \in I} A_{\xi}$  a *finite reduction* of  $\vec{b}$  if it is the image of some subsequence of  $\vec{b}$  under some orderly term over  $\mathcal{F}$ .

If  $\vec{a} \leq_{\mathcal{F}} \vec{b}$ , then each term of  $\vec{a}$  is a finite reduction of  $\vec{b}$ .

**Definition 3.4.** Suppose  $((A_{\xi})_{\xi \in I}, \mathcal{F})$  is an algebra and  $\vec{e} \in {}^{\omega}I$ , we say that the infinite sequence  $\vec{a} \in {}^{\omega}(\bigcup_{\xi \in I} A_{\xi})$  is  $\vec{e}$ -sorted if  $\vec{a}(n) \in A_{\vec{e}(n)}$  for each  $n \in \omega$ .

The reader is reminded that the family of phyla  $(A_{\xi})_{\xi \in I}$  is always assumed to be pairwise disjoint and so the sort of a sequence is well-defined, i.e. if  $\vec{b}$  is  $\vec{e}$ -sorted, then the sort is unique. If the family of sets were not pairwise disjoint and, say,  $a \in A_{\xi} \cap A_{\eta}$ ,  $\xi \neq \eta$ , then the constant sequence  $\langle a, a, \dots \rangle \in {}^{\omega}(\bigcup_{\xi \in I} A_{\xi})$  were both  $\langle \xi, \xi, \dots \rangle$ -sorted and  $\langle \eta, \eta, \dots \rangle$ -sorted.

Before proceeding to the next definition, a moment’s reflection should reveal that, when restricted to  $\vec{e}$ -sorted sequences for a fixed sort  $\vec{e}$ , the relation  $\leq_{\mathcal{F}}$  remains a preorder.

<sup>2</sup>The term “reduction” has been defined for infinite sequences in Definition 3.2. We may extend this notion to the family of finite sequences, which we do not do formally, but instead when the word is used in this context, it is understood to be as described.

*Definition 3.5.* If  $\vec{b}$  is an  $\vec{e}$ -sorted sequence, define

$$FR_{\mathcal{F}}^{\vec{e}}(\vec{b}) = \{\vec{a}(0) : \vec{a} \leq_{\mathcal{F}} \vec{b} \text{ and } \vec{a} \text{ is } \vec{e}\text{-sorted}\}.$$

When the algebra is homogeneous, we drop the reference to the sort  $\vec{e}$  since it is unique; in such setting, each of the sets  $FR_{\mathcal{F}}(\vec{b})$  coincides with the set of finite reductions of  $\vec{b}$ :

$$FR_{\mathcal{F}}(\vec{b}) = \{a \in A : a = f(\vec{b}_0), f \in OT(\mathcal{F}), \text{ and } \vec{b}_0 \text{ a finite subsequence of } \vec{b}\}.$$

*Remark 3.1.* Take note that  $FR_{\mathcal{F}}^{\vec{e}}(\vec{b})$  is a *subset* of the set of finite reductions of  $\vec{b}$  belonging to the phylum  $A_{\vec{e}(0)}$ . The  $FR$  in  $FR_{\mathcal{F}}^{\vec{e}}(\vec{b})$  might inadvertently lead the reader to think that  $FR_{\mathcal{F}}^{\vec{e}}(\vec{b})$  is the set of finite reductions of  $\vec{b}$  belonging to the phylum  $A_{\vec{e}(0)}$ ; it is so for the case of a homogeneous algebra, but it is not in general. Examples of this difference will start to show up in Section 4; see Example 5.1 in particular.

Consider again the algebra  $(\mathbb{Z}^+, \{+\})$  of Hindman's Theorem for an illustration. Every finite sum of the terms in a given sequence  $\vec{b} \in {}^\omega\mathbb{Z}^+$  is a finite reduction of the sequence. Indeed,

$$FR_{\{+\}}(\vec{b}) = \left\{ \sum_{j=1}^n \vec{b}(i_j) : \langle \vec{b}(i_1), \dots, \vec{b}(i_n) \rangle \text{ is a finite subsequence of } \vec{b} \right\}.$$

We now come to the central notion of our investigation.

*Definition 3.6 ( $\vec{e}$ -Ramsey Algebra).* Let  $\vec{e} \in {}^\omega I$ . An algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is said to be an  $\vec{e}$ -Ramsey algebra if, for each  $\vec{e}$ -sorted sequence  $\vec{b}$  and  $X \subseteq A_{\vec{e}(0)}$ , there exists an  $\vec{e}$ -sorted reduction  $\vec{a}$  of  $\vec{b}$  such that  $FR_{\mathcal{F}}^{\vec{e}}(\vec{a})$  is either contained in or disjoint from  $X$ .

Such a sequence  $\vec{a}$  is said to be *homogeneous* for  $X$  (with respect to  $\mathcal{F}$ ).

In the case where the algebra is homogeneous, an  $\vec{e}$ -Ramsey algebra is simply called a *Ramsey algebra* as the set of sorts is a singleton.

The content of Hindman's Theorem can be cast in terms of the notions discussed thus far, namely for each  $X \subseteq \mathbb{Z}^+$  and for each sequence  $\vec{b}$  of positive integers, there exists a sequence  $\vec{a} \leq_{\{+\}} \vec{b}$  such that  $FR_{\{+\}}(\vec{a}) \subseteq X$  or  $X \cap FR_{\{+\}}(\vec{a}) = \emptyset$ . This is to say that  $(\mathbb{Z}^+, \{+\})$  is a (homogeneous) Ramsey algebra. In fact, a more general theorem holds; we state it as a proposition (cf. [6], V, Sec. 2):

*Proposition 3.1.* Every semigroup is a Ramsey algebra.

Thus,  $(\mathbb{V}, +)$  is a Ramsey algebra.

We end this section with a simple fact that is usually assumed tacitly in the literature.

*Proposition 3.2.* Let  $A = \bigcup_{\xi \in I} A_\xi$ , where  $(A_\xi)_{\xi \in I}$  is a pairwise disjoint family of nonempty sets. If  $(A, \mathcal{F})$  is an algebra and  $\vec{e} \in {}^\omega I$ , then the following are equivalent:

( $\sharp$ )  $(A, \mathcal{F})$  is an  $\vec{e}$ -Ramsey algebra.

- (b) For each finite coloring of  $A_{\vec{e}(0)}$  and for each  $\vec{e}$ -sorted  $\vec{b} \in {}^\omega A$ , there exists an  $\vec{e}$ -sorted  $\vec{a} \leq_{\mathcal{F}} \vec{b}$  such that  $FR_{\mathcal{F}}^{\vec{e}}(\vec{a})$  is monochromatic.

*Proof.* (b  $\Rightarrow$  #) Special case of a 2-coloring.

(#  $\Rightarrow$  b) We prove by induction on the number of coloring.

The base case follows from #.

Suppose the conclusion of b is true for an  $n$ -coloring, where  $n \geq 2$ . Given an  $(n+1)$ -coloring of  $A_{\vec{e}(0)} = X_1 \cup \dots \cup X_{n+1}$  and an  $\vec{e}$ -sorted sequence  $\vec{b}$ , consider the  $n$ -coloring of  $A_{\vec{e}(0)} = (X_1 \cup X_2) \cup \dots \cup X_{n+1}$ . By induction hypothesis, let  $\vec{c}$  be an  $\vec{e}$ -sorted reduction of  $\vec{b}$  such that  $FR_{\mathcal{F}}^{\vec{e}}(\vec{c})$  is monochromatic with a color  $i > 2$  or  $FR_{\mathcal{F}}^{\vec{e}}(\vec{c})$  is a subset of  $X_1 \cup X_2$ . If the former case holds, we are done. Otherwise,  $FR_{\mathcal{F}}^{\vec{e}}(\vec{c}) \subseteq X_1 \cup X_2$ , in which case we let  $\vec{a}$  be an  $\vec{e}$ -sorted reduction of  $\vec{c}$  (hence a reduction of  $\vec{b}$  by transitivity) homogeneous for  $X_1$  (exists under the assumption #). Then either  $FR_{\mathcal{F}}^{\vec{e}}(\vec{a})$  is monochromatic with color 1 or monochromatic with color 2.  $\square$

#### 4. FROM RAMSEY SPACE TO RAMSEY ALGEBRA

The origin of Ramsey algebra has its roots in the notion of a Ramsey space introduced by Carlson. The subject has since been a hot topic of research; the interested reader is referred to Todorcevic [12]. This section is intended to give a short account of the connection between these two notions. Our discussion of Ramsey spaces here is tailored to suit our purpose of discussing Ramsey algebras and it is not as general as is introduced by Carlson.

A preorder is a relation that is reflexive and transitive. If  $R$  is a nonempty set of infinite sequences equipped with a preorder  $\leq$ , then for all  $\vec{b} \in R$  and all  $n \in \omega$ , the sets

$$(1) \quad [n, \vec{b}] = \{\vec{a} \in R : \vec{a} \leq \vec{b} \text{ and } \vec{a} \upharpoonright n = \vec{b} \upharpoonright n\}$$

form a neighborhood basis of the *natural topology* on  $R$ .

*Definition 4.1.* Let  $R$  be a nonempty set of infinite sequences equipped with a preorder  $\leq$  and let  $X \subseteq R$ . The set  $X$  is said to be *Ramsey* if, for each  $\vec{b} \in R$  and each  $n \in \omega$ , there exists  $\vec{a} \in [n, \vec{b}]$  such that  $[n, \vec{a}] \subseteq X$  or  $[n, \vec{a}] \subseteq X^C$ . The set  $X$  is said to be *Ramsey null* in the event there exists an  $\vec{a} \in [n, \vec{b}]$  for every given  $\vec{b}$  and  $n \in \omega$  such that  $[n, \vec{a}] \subseteq X^C$ .

*Definition 4.2.* If  $R$  is a nonempty set of infinite sequences equipped with the preorder  $\leq$ , then  $(R, \leq)$  endowed with the natural topology is called a *Ramsey space* if every set which has the Property of Baire is Ramsey and every meager set is Ramsey null<sup>3</sup>.

The neighborhoods given in Eq. 1 and the natural topology induced by these sets are defined with the preorder induced by an algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$  in mind. We recall from Section 2 that the relation  $\leq_{\mathcal{F}}$  for any given algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is a preordering on the infinite sequences of  ${}^\omega(\bigcup_{\xi \in I} A_\xi)$ ; we will be chiefly interested in the natural topology on subsets of  ${}^\omega(\bigcup_{\xi \in I} A_\xi)$  determined

<sup>3</sup>Under the Axiom of Choice, the latter property can be discarded.

by any given sort  $\vec{e}$  as these are the topologies pertaining to the notion of  $\vec{e}$ -Ramsey algebra. For each sort  $\vec{e}$ , let us define  $R^{\vec{e}}$  to be the set consisting of  $\vec{e}$ -sorted sequences of  $\bigcup_{\xi \in I} A_\xi$  equipped with the relation  $\leq_{\mathcal{F}} \upharpoonright R^{\vec{e}} \times R^{\vec{e}}$ . Keep in mind that  $\leq_{\mathcal{F}}$  restricted to  $R^{\vec{e}} \times R^{\vec{e}}$  remains a preorder; we will abuse notation and denote such restriction by the same symbol  $\leq_{\mathcal{F}}$  when no confusion arises.

Given an algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$ , we define  $\mathfrak{R}^{\vec{e}}((A_\xi)_{\xi \in I}, \mathcal{F})$  to be the structure  $(R^{\vec{e}}, \leq_{\mathcal{F}})$  for each sort  $\vec{e}$ ; equip each instance with the natural topology generated by the neighborhood basis consisting of the members  $[n, \vec{b}]^{\vec{e}} = \{\vec{a} \in R^{\vec{e}} : \vec{a} \leq_{\mathcal{F}} \vec{b} \text{ and } \vec{a} \upharpoonright n = \vec{b} \upharpoonright n\}$ . For emphasis, we add a superscript  $\vec{e}$  to the members of the neighborhood basis to indicate the sort of the sequences in question. It should be reminded that  $\mathfrak{R}^{\vec{e}}((A_\xi)_{\xi \in I}, \mathcal{F})$  is comprised of  $\vec{e}$ -sorted sequences and the same goes with the sets  $[n, \vec{b}]^{\vec{e}}$ . We simply write  $\mathfrak{R}^{\vec{e}}$  in place of  $\mathfrak{R}^{\vec{e}}((A_\xi)_{\xi \in I}, \mathcal{F})$  while working within a particular algebra  $((A_\xi)_{\xi \in I}, \mathcal{F})$ .

The next two theorems of Carlson [2], adapted to the current context, are key to relating the notion of a Ramsey algebra to that of a Ramsey space. The first theorem is somewhat an abstract Ellentuck Theorem and it breaks down topological considerations of the topic into combinatorics.

*Theorem 4.1.* Suppose  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is an algebra, where  $\mathcal{F}$  is a finite family of *nonunary* operations. For each sort  $\vec{e}$ , the structure  $\mathfrak{R}^{\vec{e}}$  is a Ramsey space if and only if  $\mathfrak{R}^{\vec{e}}$  satisfies  $\clubsuit$  for each  $n \in \omega$ :

- ( $\clubsuit$ ) whenever  $\vec{b} \in R^{\vec{e}}$  and  $X$  is a set of initial segments of length  $n + 1$  of  $\vec{e}$ -sorted sequences, then there exists  $\vec{a} \in [n, \vec{b}]^{\vec{e}}$  such that the set consisting of all the initial segments of sequences in  $[n, \vec{a}]^{\vec{e}}$  of length  $n + 1$  is either a subset of  $X$  or is disjoint from  $X$ .

We have required that the family  $\mathcal{F}$  be finite and this is often the case of algebras. Carlson's original consideration does not require  $\mathcal{F}$  to be finite as the consideration is of a more general situation; recall that Ramsey algebras intend to capture Ramsey spaces induced by algebras from a combinatorial point of view.

*Theorem 4.2.* Under the hypothesis of Theorem 4.1,  $\mathfrak{R}^{\vec{e}}$  is a Ramsey space if and only if, for each  $m \in \omega$ ,  $\mathfrak{R}^{\vec{e}-m}$  satisfies the special case of  $\clubsuit$  with  $n = 0$ .

The preceding theorem, when paraphrased, becomes a statement in the language of Ramsey algebra.

*Corollary 4.1.* Under the hypothesis of Theorem 4.1, the structure  $\mathfrak{R}^{\vec{e}}$  is a Ramsey space if and only if  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is an  $(\vec{e} - m)$ -Ramsey algebra for each  $m \in \omega$ .

In particular, if  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is homogeneous, then  $\mathfrak{R}^{\vec{e}}$  is a Ramsey space if and only if  $((A_\xi)_{\xi \in I}, \mathcal{F})$  is a Ramsey algebra.

## 5. HETEROGENEOUS RAMSEY ALGEBRAIC PROPERTIES OF VECTOR SPACES

We now begin the study of vector spaces in the context of Ramsey algebra. A vector space has a two-sorted heterogeneous nature:  $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, +_{\mathbb{F}}, \times_{\mathbb{F}}, \cdot)$ . In

this section,  $\mathcal{F}$  is reserved for the family  $\{+_{\mathbb{V}}, +_{\mathbb{F}}, \times_{\mathbb{F}}, \cdot\}$  and we always let  $A_0$  denote the underlying field of a vector space and  $A_1$  the set of vectors.

A few observations regarding the orderly terms of  $\mathcal{F}$  will come in handy.

*Lemma 5.1.* Let  $(\mathbb{V}, \mathbb{F}, \mathcal{F})$  be a vector space and  $f \in OT(\mathcal{F})$ . The following hold:

- (1) The only unary orderly terms over  $\mathcal{F}$  are the identity functions  $id_{\mathbb{F}}$  and  $id_{\mathbb{V}}$ ; the only binary orderly terms over  $\mathcal{F}$  are the binary operations in  $\mathcal{F}$ .
- (2) If  $Dom(f) = \mathbb{F}^n$  for some nonzero  $n \in \omega$ , then  $Rn(f) \subseteq \mathbb{F}$ .
- (3) If  $Dom(f) = A_{\xi_1} \times \cdots \times A_{\xi_n}$  for some natural number  $n \geq 1$  and there exist  $i \leq n$  such that  $\xi_i = 1$ , then it follows that  $Rn(f) \subseteq \mathbb{V}$ .
- (4) If  $f$  satisfies the hypothesis of Fact 3, then  $\xi_n = 1$  (i.e.  $A_{\xi_n} = \mathbb{V}$ ).
- (5) If  $f$  is vector valued and all the scalar components of  $\bar{x}$  have the value 0, then  $f(\bar{x})$  is either the zero vector or the sum of some vectors appearing in  $\bar{x}$ ; if  $f$  is scalar valued, then  $f(\bar{x}) = 0$ .

*Proof.* It takes some simple induction on the generation of the orderly terms to establish Facts 1 and 2.

Facts 3 through 5 can also be established by induction on the generation of orderly terms. We will only give the proof of Fact 4 and a sketch of the proof of Fact 5.

Fact 4: The implication holds if  $f$  is either  $id_{\mathbb{F}}$  or  $id_{\mathbb{V}}$ . Similarly, if  $f \in \mathcal{F}$ , then the antecedent of the implication holds provided  $f$  is either scalar multiplication or vector addition. In both cases, the consequent of the implication holds.

Now, suppose  $f(\bar{x}_1, \bar{x}_2) = g(h_1(\bar{x}_1), h_2(\bar{x}_2))$  for some  $g \in \mathcal{F}$  and orderly terms  $h_1$  and  $h_2$ , where the arity of  $g$  owes to the fact that every  $g \in \mathcal{F}$  is binary. If  $g$  is field addition or field multiplication, then both  $h_1$  and  $h_2$  are scalar-valued and hence  $\bar{x}_1$  and  $\bar{x}_2$  are all scalar terms, contradicting  $\xi_n = 1$ . Thus,  $g$  is either vector addition or scalar multiplication. In both cases,  $h_2$  is vector valued. Hence,  $\bar{x}_2$  involves some vector term. Then, by the induction hypothesis, the trailing term of  $\bar{x}_2$  is a vector.

Fact 5: For the base cases, the conclusion is obvious. Thus, suppose  $f(\bar{x}) = g(h_1(\bar{x}_1), h_2(\bar{x}_2))$  with  $g \in \mathcal{F}$  and with  $h_1, h_2 \in OT(\mathcal{F})$  both satisfying the conclusion of Fact 5. Now assume  $\bar{x}$  satisfies the hypothesis of Fact 5. If  $g$  is vector addition, then  $f(\bar{x}) = h_1(\bar{x}_1) + h_2(\bar{x}_2)$ , and so  $f$  clearly satisfies the conclusion of Fact 5; this is the case where the result is the nonempty sum of some vectors from  $\bar{x}$ . Proofs of the other cases of  $g$  are similarly obvious and will, hence, be omitted.  $\square$

The next lemma will be helpful in proving various Ramsey algebraic properties of vector spaces. Observe that as  $\vec{e}$  is infinite and  $I = \{0, 1\}$  is finite, either 0, 1, or both must occur infinitely many times in any given sort  $\vec{e}$ . There are three cases as such, namely  $\vec{e}$  being eventually constant with either value, or  $\vec{e}$  consisting infinitely many of both 0's and 1's.

*Lemma 5.2.* Let  $(\mathbb{V}, \mathbb{F}, \mathcal{F})$  be a vector space, let  $\vec{b}$  be an  $\vec{e}$ -sorted sequence of elements of  $\mathbb{F} \cup \mathbb{V}$ , and let  $\vec{a}$  be an  $\vec{e}$ -sorted reduction of  $\vec{b}$ . If  $\vec{e}$  is eventually



constant with  $n^* \in \omega$  being *least* such that  $\vec{e}-(n^*+1)$  is constant, then  $\vec{a}(i) = \vec{b}(i)$  for each  $i \leq n^*$ .

*Proof.* The proofs of both cases of eventually constant sort  $\vec{e}$  are similar and we will only provide for the eventually constant with value 0 case. In such a case,  $\vec{e}(n^*) = 1$  and  $\vec{e}(i) = 0$  for all  $i > n^*$ .

To begin the proof, for each  $i \leq n^*$ , let  $\vec{b}_i$  and  $f_i \in \text{OT}(\mathcal{F})$  be such that  $\vec{a}(i) = f_i(\vec{b}_i)$  and  $\vec{b}_1 * \dots * \vec{b}_{n^*}$  is a subsequence of  $\vec{b}$ . Let the leading term of  $\vec{b}_{n^*}$  be  $\vec{b}(j)$ . We claim that  $n^* = j$ . Clearly,  $n^* \leq j$ . Now assume that  $n^* < j$ . Then, we would have  $f_{n^*}$  operating on scalars, yielding a scalar (Fact 2, Lemma 5.1), contradicting  $\vec{a}(n^*)$  being a vector. That  $f_{n^*}$  is unary follows from Fact 4 of Lemma 5.1.

It now follows by counting pigeonholes that  $f_i$  for each  $i < n^*$  is also unary, consequently each of the  $f_i$  for  $i \leq n^*$ , is an identity function. Clearly, the equality  $\vec{a}(i) = \vec{b}(i)$  holds for each  $i \leq n^*$ .  $\square$

We insert an interlude to provide an example promised earlier.

*Example 5.1.* Let us consider the vector space consisting of real functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  over the real numbers, which we let the boldface  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\mathbf{2}$  be the constant function mapping all reals to the numbers 0, 1, 2, respectively. Consider the infinite sequence  $\vec{b} = \langle \mathbf{1}, 0, \mathbf{1}, 0, 0, 0, \dots \rangle$  of sort  $\vec{e}$  satisfying  $\vec{e}(i) = 0$  for all  $i \in \omega$  except for  $i = 0, 2$ . Then the set of finite reductions is  $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, 0\}$ ; however,  $FR_{\mathcal{F}}^{\vec{e}}(\vec{b}) = \{\mathbf{1}\}$  by the lemma above.

Our discussion thus far culminates into a corollary to Lemma 5.2:

*Proposition 5.1.* If  $\vec{e}$  is eventually constant except when  $\vec{e} = \langle 0, 0, 0, \dots \rangle$ , then every vector space is an  $\vec{e}$ -Ramsey algebra. Every vector space is a  $\langle 0, 0, 0, \dots \rangle$ -Ramsey algebra provided the underlying field is finite.

The case where  $\vec{e}$  has infinitely many of both scalars and vectors is less trivial. The first proposition for such case concerns vector spaces over finite fields.

*Proposition 5.2.* Every vector space over a finite field is an  $\vec{e}$ -Ramsey algebra for sorts  $\vec{e}$  with infinitely many 0's and infinitely many 1's.

*Proof.* Fix  $\vec{e}$ . Let  $X \subseteq A_{\vec{e}(0)}$  be given and let  $\vec{b}$  be an  $\vec{e}$ -sorted sequence of elements of  $\mathbb{F} \cup \mathbb{V}$ . Since there are infinitely many scalars in  $\vec{b}$  and the field  $\mathbb{F}$  is finite, there exists a scalar, call it  $\rho$ , which occurs infinitely often in  $\vec{b}$ . The scalar  $\rho$  has a finite order, meaning there exists an  $s \in \mathbb{Z}^+$  such that  $s\rho = 0$ .

We construct an  $\vec{e}$ -sorted reduction  $\vec{a}$  of  $\vec{b}$  recursively; we will be applying the remark after Definition 3.2 to ensure our construction is indeed a reduction by requiring every initial segment of  $\vec{a}$  be a reduction of an initial segment of  $\vec{b}$ .

(Case  $\vec{e}(0) = 0$ .) Set  $\vec{a}(0) = 0$ . Suppose  $\langle \vec{a}(0), \dots, \vec{a}(N) \rangle$  has been constructed as a reduction of some initial segment  $\sigma$  of  $\vec{b}$ . If  $\vec{e}(N+1) = 0$ , set  $\vec{a}(N+1) = 0$ . On the other hand, if  $\vec{e}(N+1) = 1$ , then let  $\vec{a}(N+1) = v$ , where  $v$  is the first vector in  $\vec{b} - \sigma$ . Notice that  $\langle \vec{a}(0), \dots, \vec{a}(N+1) \rangle$  is a reduction of some initial segment  $\vec{b}$  extending  $\sigma$  (in the case  $\vec{e}(N+1) = 0$ , we use the fact that there are infinitely many, and thus at least  $s$  many,  $\rho$ 's in  $\vec{b} - \sigma$ ).

Now, since all scalar terms are 0 in  $\vec{a}$ , by Facts 3 and 5 of Lemma 5.1, it follows that we have  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) = \{0\}$ . Clearly,  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X$  or  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X^C$ , and the proposition for the case  $\vec{e}(0) = 0$  follows.

(Case  $\vec{e}(0) = 1$ .) Our plan is to start from the given  $\vec{b}$  and obtain  $\vec{a} \leq_{\mathcal{F}} \vec{c} \leq_{\mathcal{F}} \vec{b}$  of the same sort  $\vec{e}$  so that  $\vec{a}$  is homogeneous for  $X$ . In the process, we will make use of an auxiliary sequence  $\vec{a}'$ .

The first step is, in a manner similar to the previous case, to reduce  $\vec{b}$  to the sequence  $\vec{c}$  having the property that its scalar terms are all the zero scalar.

The next step is to obtain an auxiliary sequence  $\vec{a}'$  homogeneous for  $X$  with respect to  $\{+_{\mathbb{V}}\} \subseteq \mathcal{F}$ , obtained by considering the subsequence of vector terms of  $\vec{c}$  and applying Proposition 3.1 (recall that  $(\mathbb{V}, +_{\mathbb{V}})$  is a semigroup).

To obtain  $\vec{a}$ , set  $\vec{a}(0) = \vec{a}'(0)$ . Suppose  $\langle \vec{a}(0), \dots, \vec{a}(N) \rangle$  has been constructed from  $\vec{c}$  as a reduction of an initial segment  $\sigma$  of  $\vec{c}$  and such that the subsequence of vector terms of  $\langle \vec{a}(0), \dots, \vec{a}(N) \rangle$  forms a subsequence of  $\vec{a}'$ . Now, if  $\vec{e}(N+1) = 0$ , set  $\vec{a}(N+1) = 0$ . On the other hand, if  $\vec{e}(N+1) = 1$ , set  $\vec{a}(N+1) = \vec{a}'(N')$  in such a way that  $N'$  is the least such that  $\vec{a}'(N')$  is a finite reduction of  $\vec{c} - \sigma$  (possible because  $\vec{a}'$  is a reduction of  $\vec{c}$ ) and such that the subsequence of vector terms of  $\langle \vec{a}(0), \dots, \vec{a}(N), \vec{a}'(N') \rangle$  forms a subsequence of  $\vec{a}'$ . This extended sequence is yet another reduction of an extended initial segment of  $\vec{c}$ . Thus,  $\vec{a}$  is an  $\vec{e}$ -sorted reduction of  $\vec{c}$ .

To recapitulate the essential properties of  $\vec{a}$ , we have that all the scalar terms of  $\vec{a}$  are 0, whereas the vector terms form a subsequence of  $\vec{a}'$ . Thus, by Fact 5 of Lemma 5.1, if  $f \in \text{OT}(\mathcal{F})$  is vector-valued and  $\vec{q}$  is a finite subsequence of  $\vec{a}$ , then  $f(\vec{q})$  is a sum of some vectors in  $\vec{q}$ . It follows by the homogeneity of  $\vec{a}'$  with respect to  $X$  that  $f(\vec{q}) \in X$  for all such  $\vec{q}$  and  $f \in \text{OT}(\mathcal{F})$ , or  $f(\vec{q}) \notin X$  for all such  $\vec{q}$  and  $f \in \text{OT}(\mathcal{F})$ . Therefore,  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X$  or  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \subseteq X^C$  owing to the fact stated in Remark 3.1. Thus, given  $\vec{b}$  and  $X \subseteq A_{\vec{e}(0)}$ , there exists  $\vec{e}$ -soretd  $\vec{a} \leq_{\mathcal{F}} \vec{b}$  such that  $\vec{a}$  is homogeneous for  $X$  with respect to  $\mathcal{F}$ .  $\square$

The story is different in the case  $\mathbb{F}$  is infinite (look ahead to Proposition 5.3). We build up the proof of this fact with two lemmas.

*Lemma 5.3.* Suppose  $(\mathbb{F}, +_{\mathbb{F}}, \times_{\mathbb{F}})$  is an infinite field. There exists a sequence  $\vec{\beta} \in {}^{\omega}\mathbb{F}$  such that for every orderly terms  $f, g, f', g'$  over  $\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}$  and for every finite subsequences  $\vec{\beta}_0 * \vec{\beta}_1$  and  $\vec{\beta}_2 * \vec{\beta}_3$  of  $\vec{\beta}$ , the inequation below holds:

$$(2) \quad f(\vec{\beta}_0) +_{\mathbb{F}} g(\vec{\beta}_1) \neq f'(\vec{\beta}_2) \times_{\mathbb{F}} g'(\vec{\beta}_3).$$

Such sequence  $\vec{\beta}$  exists owing really to the growth rate of the orderly terms over the field operations. To construct  $\vec{\beta}$  recursively, we can begin with two distinct field elements almost arbitrarily. The orderly terms that can operate on these elements are limited to the field operations themselves. To construct the third element, we only need to pick a field element outside the image set of the first two elements under the field operations with an added requirement, namely the element picked must satisfy the desired inequation (2). Such procedure is carried out *ad infinitum* where, at each step, the next element is picked from outside the image set of the previous elements under the orderly terms that can operate on them, the number of available orderly terms which is finite, and

explicitly requiring the inequation (2) to hold thus far. Each recursive step can proceed because of the assumption that the field is *infinite*. For a complete proof, see Lemma 5.4 of [9].

*Corollary 5.1.* No infinite field is a Ramsey algebra.

The proof of this corollary hinges upon a sequence  $\vec{\beta}$  given in Lemma 5.3 and the associated subset  $Y$  of the field  $\mathbb{F}$ :

$$(3) \quad Y = \{f(\vec{\beta}_0) +_{\mathbb{F}} g(\vec{\beta}_1) : \Psi(f, g)\}$$

where  $\Psi(f, g)$  is the statement “ $f, g \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$  and  $\vec{\beta}_0 * \vec{\beta}_1$  is a finite subsequence of  $\vec{\beta}$ .”

A moment’s reflection reveals that, for each  $f, g \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$  and each finite subsequence  $\vec{\beta}_0 * \vec{\beta}_1$  of  $\vec{\beta}$ , we have

$$f(\vec{\beta}_0) \times_{\mathbb{F}} g(\vec{\beta}_1) \notin Y.$$

Thus, given an arbitrary  $\vec{a} \leq_{\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}} \vec{\beta}$ , we have  $\vec{a}(0) +_{\mathbb{F}} \vec{a}(1) \in Y$  and  $\vec{a}(0) \times_{\mathbb{F}} \vec{a}(1) \notin Y$ . Hence, no  $\vec{a} \leq_{\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}} \vec{\beta}$  is homogeneous for  $Y$ , whence the corollary follows.

*Lemma 5.4.* Let  $\mathbb{F}$  be infinite and  $\vec{e}$  a given sort. Suppose  $\vec{\beta} \in {}^{\omega}\mathbb{F}$  and suppose  $v \in \mathbb{V}$  is a fixed *nonzero* vector. Define

$$\vec{b}(i) = \begin{cases} \vec{\beta}(i) & \text{if } \vec{e}(i) = 0 \\ \vec{\beta}(i) \cdot v & \text{otherwise.} \end{cases}$$

Then the following holds:

If  $F$  is an orderly term over  $\mathcal{F}$  and  $\vec{b}_0$  is some finite subsequence of  $\vec{b}$ , then there exists some  $|\vec{b}_0|$ -ary orderly term  $f$  over  $\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}$  such that

$$F(\vec{b}_0) = \begin{cases} f(\vec{b}_0) & \text{if } \text{Rn}(F) \subseteq \mathbb{F} \\ f(\vec{b}_0) \cdot v & \text{if } \text{Rn}(F) \subseteq \mathbb{V} \end{cases}.$$

*Proof.* When  $\text{Rn}(F) \subseteq \mathbb{F}$ , Lemma 5.1 implies that  $\text{Dom}(F)$  must be the Cartesian product  $\mathbb{F}^n$ . We may let  $f = F$ .

As for the case  $\text{Rn}(F) \subseteq \mathbb{V}$ , we prove the conclusion of the lemma by induction on the length of the subsequence  $\langle \vec{b}(i_1), \dots, \vec{b}(i_n) \rangle$ . Starting from  $n = 1$ ,

$$F(\vec{b}(i_1)) = \text{id}_{\mathbb{V}}(\vec{b}(i_1)) = \text{id}_{\mathbb{V}}(\vec{\beta}(i_1) \cdot v) = \vec{\beta}(i_1) \cdot v = \text{id}_{\mathbb{F}}(\vec{\beta}(i_1)) \cdot v,$$

by Fact 1 of Lemma 5.1 and clearly  $\text{id}_{\mathbb{F}} \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$ .

For the inductive step, let  $F$  be of arity  $N + 1$  and  $\langle \vec{b}(i_1), \dots, \vec{b}(i_{N+1}) \rangle$  some subsequence of  $\vec{b}$  of length  $N + 1$ . Dismantling  $F$  into its components gives

$$F(\vec{b}(i_1), \dots, \vec{b}(i_{N+1})) = g(h_1(\vec{b}(i_1), \dots, \vec{b}(i_r)), h_2(\vec{b}(i_{r+1}), \dots, \vec{b}(i_{N+1}))),$$

for some  $g \in \mathcal{F}$  and  $h_1, h_2 \in \text{OT}(\mathcal{F})$ . Either  $g$  is vector addition or  $g$  is scalar multiplication ( $g$  cannot be a field operation because  $\text{Rn}(f) \subseteq \mathbb{V}$  by case assumption).

If  $g$  is vector addition, then  $h_1$  and  $h_2$  are vector-valued. The induction hypothesis applies to give

$$\begin{aligned} & F(\vec{b}(i_1), \dots, \vec{b}(i_{N+1})) \\ &= (f_1(\vec{\beta}(i_1), \dots, \vec{\beta}(i_r)) \cdot v) +_{\mathbb{V}} (f_2(\vec{\beta}(i_{r+1}), \dots, \vec{\beta}(i_{N+1})) \cdot v) \\ &= (f_1(\vec{\beta}(i_1), \dots, \vec{\beta}(i_r)) +_{\mathbb{F}} f_2(\vec{\beta}(i_{r+1}), \dots, \vec{\beta}(i_{N+1}))) \cdot v \end{aligned}$$

for some  $f_1, f_2 \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$ . The required  $f \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$  can be clearly seen to be  $f(x_1, \dots, x_{N+1}) = f_1(x_1, \dots, x_r) +_{\mathbb{F}} f_2(x_{r+1}, \dots, x_{N+1})$ .

On the other hand, if  $g$  is scalar multiplication, then  $h_1$  is scalar-valued and  $h_2$  is vector-valued, and the induction hypothesis applies to give

$$F(\vec{b}(i_1), \dots, \vec{b}(i_{N+1})) = (f_1(\vec{\beta}(i_1), \dots, \vec{\beta}(i_r))) \cdot (f_2(\vec{\beta}(i_{r+1}), \dots, \vec{\beta}(i_{N+1})) \cdot v)$$

for some  $f_1, f_2 \in \text{OT}(\{+_{\mathbb{F}}, \times_{\mathbb{F}}\})$ , from which it follows that

$$F(\vec{b}(i_1), \dots, \vec{b}(i_{N+1})) = f(\vec{b}(i_1), \dots, \vec{b}(i_{N+1})) \cdot v,$$

where  $f(x_1, \dots, x_{N+1}) = f_1(x_1, \dots, x_r) \times_{\mathbb{F}} f_2(x_{r+1}, \dots, x_{N+1})$  is again an orderly term over  $\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}$ .  $\square$

A vector space is nontrivial if  $\mathbb{V} \neq \{O\}$ .

*Proposition 5.3.* No nontrivial vector space over an infinite field is an  $\vec{e}$ -Ramsey algebra if  $\vec{e}$  is a sort with infinitely many of both 0's and 1's.

*Proof.* Let  $\vec{\beta}$  be a fixed sequence guaranteed by Lemma 5.3 and let  $Y$  be the associated set given in Eq. 3. Pick a nonzero  $v \in V$  and define  $\vec{b}$  by

$$\vec{b}(i) = \begin{cases} \vec{\beta}(i) & \text{if } \vec{e}(i) = 0 \\ \vec{\beta}(i) \cdot v & \text{otherwise.} \end{cases}$$

Accompanying the set  $Y$  is the set  $X = \{\alpha \cdot v : \alpha \in Y\}$ .

If  $\vec{a} \leq_{\mathcal{F}} \vec{b}$ , an application of the preceding lemma then shows that for each  $i \in \omega$ , there exist some orderly term  $h_i$  over  $\{+_{\mathbb{F}}, \times_{\mathbb{F}}\}$  and some finite subsequence  $\vec{\beta}_0$  of  $\vec{\beta}$  such that

$$\vec{a}(i) = \begin{cases} h_i(\vec{\beta}_0) & \text{if } \vec{e}(i) = 0 \\ h_i(\vec{\beta}_0) \cdot v & \text{otherwise.} \end{cases}$$

Fix an arbitrary  $\vec{e}$ -sorted reduction  $\vec{a}$  of  $\vec{b}$ . It suffices to show that  $\vec{a}$  is not homogeneous with respect to  $X$  or  $Y$  depending on whether  $\vec{e}(0) = 0$  or  $\vec{e}(0) = 1$ , respectively:

(When  $\vec{e}(0) = 1$ .) First, pick  $m_2 > m_1 > 0$  such that  $\vec{e}(m_1) = 0$  and  $\vec{e}(m_2) = 1$ . Let  $\vec{c}_1, \vec{c}_2$  be  $\vec{e}$ -sorted reductions of  $\vec{a}$  such that

$$\begin{aligned} \vec{c}_1(0) &= \vec{a}(0) +_{\mathbb{V}} \vec{a}(m_2) \\ &= h_0(\vec{\beta}_0) \cdot v +_{\mathbb{V}} h_{m_2}(\vec{\beta}_1) \cdot v \\ &= [h_0(\vec{\beta}_0) +_{\mathbb{F}} h_{m_2}(\vec{\beta}_1)] \cdot v, \end{aligned}$$

whereas

$$\begin{aligned}
\vec{c}_2(0) &= \vec{a}(m_1) \cdot \vec{a}(m_2) \\
&= h_{m_1}(\vec{\beta}_0) \cdot [h_{m_2}(\vec{\beta}_1) \cdot v] \\
&= [h_{m_1}(\vec{\beta}_0) \times_{\mathbb{F}} h_{m_2}(\vec{\beta}_1)] \cdot v.
\end{aligned}$$

(Such  $\vec{c}_1$  and  $\vec{c}_2$  exist as  $\vec{e}$ -sorted reductions of  $\vec{a}$  since there are infinitely many of both 0's and 1's among the terms of  $\vec{e}$ : each of  $\vec{c}_1 - 1$  and  $\vec{c}_2 - 1$  forms a subsequence of  $\vec{a} - (m_2 + 1)$ .) It follows that  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \cap X \neq \emptyset$  and  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \cap X^C \neq \emptyset$  since  $\vec{c}_1(0) \in X$  while  $\vec{c}_2(0) \in X^C$ .

(When  $\vec{e}(0) = 0$ .) Pick an  $m > 0$  such that  $\vec{e}(m) = 0$ . Let  $\vec{c}_1, \vec{c}_2$  be  $\vec{e}$ -sorted reductions of  $\vec{a}$  such that  $\vec{c}_1(0) = \vec{a}(0) +_{\mathbb{F}} \vec{a}(m)$ , while  $\vec{c}_2 = \vec{a}(0) \times_{\mathbb{F}} \vec{a}(m)$ . It then follows that  $\vec{c}_1(0) \in Y$  while  $\vec{c}_2(0) \notin Y$ . As such,  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \cap Y \neq \emptyset$  and  $\text{FR}_{\mathcal{F}}^{\vec{e}}(\vec{a}) \cap Y^C \neq \emptyset$ .  $\square$

We dovetail the results above into a classification of vector spaces, thus rounding up the discussion of this section:

*Theorem 5.1.* Let  $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, +_{\mathbb{F}}, \times_{\mathbb{F}}, \cdot)$  be a vector space.

- (1) In the case where  $\mathbb{F}$  is a finite field, the vector space is an  $\vec{e}$ -Ramsey algebra for all sorts  $\vec{e}$ .
- (2) In the case where  $\mathbb{F}$  is an infinite field, the vector space is *not* an  $\vec{e}$ -Ramsey algebra for all sorts  $\vec{e}$  except when  $\vec{e}$  is a nonconstant eventually constant sequence or when it is constant with value 1.

## 6. A RELATED HOMOGENEOUS STRUCTURE

Scalar multiplication in a vector space  $(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, +_{\mathbb{F}}, \times_{\mathbb{F}}, \cdot)$  gives rise to a collection of unary functions given by  $f_r(v) = r \cdot v$  for each scalar  $r \in \mathbb{F}$  and vector  $v \in \mathbb{V}$ . We may, therefore, consider the algebra  $(\mathbb{V}, \mathcal{F})$ , where  $\mathcal{F}$  consists of the vector operation  $+$  and each of the functions  $f_r$ . Notice that the function  $f_0$  induced by the zero scalar sends every vector to the zero vector  $O$ . As a consequence, if  $\vec{b}$  is an infinite sequence of vectors, we can trivially obtain the infinite sequence consisting of the zero vector as a reduction. The implication of this observation is that  $(\mathbb{V}, \mathcal{F})$  is trivially a Ramsey algebra.

To remove the triviality brought about by the zero scalar, we consider the algebra  $(\mathbb{V}, \mathcal{K})$ , where  $\mathcal{K} = \mathcal{F} \setminus \{f_0\}$ . The symbol  $\mathcal{K}$  will be reserved for this class of functions in this section.

We proceed with a discussion of the orderly terms over  $\mathcal{K}$ . First, note that the composition  $h(x, y) = +(f_r(x), f_s(y))$  is an orderly term. The same is true if we have  $n$  functions composed iteratively in this fashion. In particular, if  $r_1, r_2, \dots, r_n$  are nonzero scalars, then

$$(4) \quad h(x_1, \dots, x_n) = +(\dots + ((f_{r_1}(x_1), f_{r_2}(x_2)), f_{r_3}(x_3)), \dots), f_{r_n}(x_n))$$

$$(5) \quad = \sum_{i=1}^n r_i \cdot x_i$$

is an orderly term over  $\mathcal{K}$ . The first lemma of this section states that orderly terms over  $\mathcal{K}$  are of the form given by Eq. 5.

*Lemma 6.1.*  $f \in OT(\mathcal{K})$  if and only if  $f(x_1, \dots, x_n) = \sum_{i=1}^n r_i \cdot x_i$  for some nonzero scalars  $r_1, \dots, r_n$ .

*Proof.* We have seen that  $\sum_{i=1}^n r_i \cdot x_i$  defines an orderly term over  $\mathcal{K}$ . As for the converse, we do induction on the generation of the orderly terms over  $\mathcal{K}$ . Thus, first observe that the claim is clearly true for the orderly term  $f_r$  of any nonzero scalar  $r$ ; it is also true when  $f$  is the identity function since  $\text{id}_{\mathbb{V}} = f_1$ . The conclusion is equally trivial for vector addition.

Now, suppose  $g \in \mathcal{K}$  and  $h_1, h_2$  are orderly terms satisfying Eq. 5, i.e. for some nonzero scalars  $r_i, s_j, i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ ,

$$\begin{aligned} h_1(x_1, \dots, x_n) &= \sum_{i=1}^n r_i \cdot x_i, \\ h_2(y_1, \dots, y_m) &= \sum_{i=1}^m s_i \cdot y_i. \end{aligned}$$

If  $f(x_1, \dots, x_n, y_1, \dots, y_m) = g(h_1(x_1, \dots, x_n), h_2(y_1, \dots, y_m))$ , where  $g$  is vector addition, then  $f(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{i=1}^n r_i \cdot x_i + \sum_{i=1}^m s_i \cdot y_i$ , which is clearly of the desired form. On the other hand, given that  $f(x_1, \dots, x_n) = f_r(h_1(x_1, \dots, x_n))$  for some nonzero scalar  $r$ , we are led to  $f(x_1, \dots, x_n) = r \cdot \sum_{i=1}^n r_i x_i = \sum_{i=1}^n (rr_i) \cdot x_i$  and none of the scalars  $rr_i$  is 0. Hence, such  $f$  also assumes the form of Eq. 5. This concludes the proof of the lemma.  $\square$

A short discussion on linear independence is instructive. Suppose  $u_1, \dots, u_n$  are linearly independent vectors and  $v = r_1 \cdot u_1 + \dots + r_n \cdot u_n$ , then it is easy to check that if  $f$  is an orderly term over  $\mathcal{K}$  and  $f(u_1, \dots, u_n) = v$ , then  $f$  is *unique*, namely  $f(x_1, \dots, x_n) = \sum_{i=1}^n r_i x_i$ . We stress that this uniqueness is not met should the vectors  $u_1, u_2, \dots, u_n$  be linearly dependent. It is also noteworthy that, as a consequence of this uniqueness, the zero vector  $O$  is not the image of any orderly term over  $\mathcal{K}$  operated on a tuple consisting of linearly independent vectors.

The first result in this section concerns the structure  $(\mathbb{V}, \mathcal{K})$  for finite dimensional vector space.

*Theorem 6.1.*  $(\mathbb{V}, \mathcal{K})$  is a Ramsey algebra for every finite dimensional vector space.

*Proof.* Let the vector space be  $n$ -dimensional and let  $\vec{b} \in {}^\omega \mathbb{V}$ . The proof is by recursion; we will wave our hand on this proof.

Since  $\vec{b}(0), \vec{b}(1), \dots, \vec{b}(n)$  are  $n+1$  vectors, they are linearly dependent and so for some scalars  $r_0, r_1, \dots, r_n$ , not all of which are 0, we obtain  $\sum_{i=0}^n r_i \cdot \vec{b}(i) = O$  (this need not correspond to an orderly term because, if it did, then none of the scalars could be 0). If  $i_0 < \dots < i_k, k \leq n$ , are indices corresponding to the nonzero scalars, then we have  $\sum_{j=0}^k r_{i_j} \cdot \vec{b}(i_j) = O$  as the image of  $(\vec{b}(i_0), \dots, \vec{b}(i_k))$  under the corresponding orderly term.

We repeat the above procedure on the subsequent  $n+1$  vectors recursively using appropriate nonzero scalars. This generates a sequence of zero vectors  $O$  as a reduction of  $\vec{b}$ , which is homogeneous for any given  $X \subseteq \mathbb{V}$ .  $\square$

The situation is different with infinite dimensional vector spaces, except for the case where the underlying field is  $\mathbb{F}_2$ . To obtain our next theorem, we will be applying Corollary 4.3 of [9]; we state the result here for convenience.

*Lemma 6.2.* Suppose  $\mathcal{H}$  is a collection of unary operations of a set  $A$ ,  $\mathcal{G}$  a collection of nonunary operations on  $A$ , and  $S$  is the set  $\{a \in A : f(a) = a \text{ for each } f \in \mathcal{H}\}$ . If  $(A, \mathcal{G})$  is a Ramsey algebra, then  $(A, \mathcal{H} \cup \mathcal{G})$  is a Ramsey algebra if and only if, for each  $\vec{b} \in {}^\omega A$ , there exists  $\vec{a} \in {}^\omega A$  such that  $\vec{a} \leq_{\mathcal{H} \cup \mathcal{G}} \vec{b}$  and  $\text{FR}_{\mathcal{G}}(\vec{a}) \subseteq S$ .

For our purpose, the set  $A$  stated in the lemma will be the set  $\mathbb{V}$  of vectors in question and  $\mathcal{H} = \{f_r : r \in \mathbb{F}, r \neq 0\}$ ,  $\mathcal{G} = \{+\}$  (thus  $\mathcal{K} = \mathcal{H} \cup \mathcal{G}$ ). It is easy to verify that the set  $S = \{v \in \mathbb{V} : f(v) = v \text{ for all } f \in \mathcal{H}\}$  of fixed points of  $\mathcal{H}$  is either the whole domain  $\mathbb{V}$  or the singleton  $\{0\}$ , depending on whether the underlying field is respectively  $\mathbb{F}_2$  or not.

*Theorem 6.2.* For no infinite dimensional vector space over  $\mathbb{F} \neq \mathbb{F}_2$  is  $(\mathbb{V}, \mathcal{K})$  a Ramsey algebra.

*Proof.* First, note that  $(\mathbb{V}, \mathcal{G})$  is a Ramsey algebra since it is a group.

We let  $\vec{b} = \langle u_0, u_1, \dots \rangle$ , where  $u_0, u_1, \dots$  are linearly independent vectors. Recall that the zero vector is not the image of any orderly term over  $\mathcal{K}$  operated on a tuple of linearly independent vectors. As such, for any  $\vec{a} \leq_{\mathcal{K}} \vec{b}$ , the element  $\vec{a}(0)$ , which is not the zero vector, is not in  $S$  and hence  $\text{FR}_{\mathcal{G}}(\vec{a}) \not\subseteq S$  for any such  $\vec{a}$ . This, by Lemma 6.2, means that the algebra is not a Ramsey algebra.  $\square$

A direct proof can also be given. Note that an algebra is not a Ramsey algebra when there exists  $\vec{b} \in {}^\omega A$  and  $X \subseteq A$  such that, for every  $\vec{a} \leq_{\mathcal{F}} \vec{b}$ , we have  $X \cap \text{FR}_{\mathcal{F}}(\vec{a}) \neq \emptyset$  as well as  $X^C \cap \text{FR}_{\mathcal{F}}(\vec{a}) \neq \emptyset$ .

The intuition is to pick out a fine bit of vectors to form  $X$ . Again, let  $\vec{b} = \langle u_0, u_1, \dots \rangle$ , where  $u_0, u_1, \dots$  are linearly independent vectors. Recall from Lemma 6.1 that for each  $f \in \text{OT}(\mathcal{K})$ , we have  $f(x_1, \dots, x_n) = \sum_{i=1}^n r_i \cdot x_i$  for some nonzero scalars  $r_1, \dots, r_n$ . Let  $X$  be defined by

$$(6) \quad X = \{v \in \mathbb{V} : \Phi(v)\},$$

where  $\Phi(v)$  is the statement “ $v = u_{i_0} + \sum_{j=1}^n r_j \cdot u_{i_j}$ ,  $r_1, \dots, r_n$  are some nonzero scalars, and  $i_0 < \dots < i_n$ .”

Given  $\vec{a} \leq_{\mathcal{K}} \vec{b}$ , we have  $\vec{a}(0) = \sum_{j=0}^n r_j \cdot u_{i_j}$  for some  $i_0 < \dots < i_n$  by Lemma 6.1. If  $r_0 = 1$ , then  $\vec{a}(0) \in X \cap \text{FR}_{\mathcal{K}}(\vec{a})$ , while a simple application of  $f_2$  would put  $f_2(\vec{a}(0)) \in X^C \cap \text{FR}_{\mathcal{K}}(\vec{a})$ . On the other hand, if  $r_0 \neq 1$ , then we have  $\vec{a}(0) \in X^C \cap \text{FR}_{\mathcal{K}}(\vec{a})$ , whereas  $f_{r_0^{-1}}(\vec{a}(0)) \in X \cap \text{FR}_{\mathcal{K}}(\vec{a})$ . In either case, we have  $X \cap \text{FR}_{\mathcal{K}}(\vec{a}) \neq \emptyset$  and  $X^C \cap \text{FR}_{\mathcal{K}}(\vec{a}) \neq \emptyset$ . Thus, the conclusion of the theorem holds.

Finally, we also have the following result for vector spaces, finite or infinite, over the field  $\mathbb{F}_2$ .

*Theorem 6.3.*  $(\mathbb{V}, \mathcal{K})$  is a Ramsey algebra for every vector space over the field  $\mathbb{F}_2$ .

*Proof.* Note that when  $\mathbb{F} = \mathbb{F}_2$ , we have  $\mathcal{K} = \{+, f_1\} = \{+, \text{id}_{\mathbb{V}}\}$ , whence  $\text{OT}(\mathcal{K})$  is the same as  $\text{OT}(\{+\})$ . The theorem follows by the fact that  $(\mathbb{V}, +)$  is a semigroup, hence a Ramsey algebra (Proposition 3.1).  $\square$

## 7. CONCLUSION

Much of the algebraic and analytical aspects of vector spaces have been studied and utilized ever since the notion of a vector space came into prominence in mathematics and science. This paper offers a glimpse at the less studied combinatorics of vector spaces in the context of a combinatorial notion which is at its infancy stage.

We wrap up with a summary of the results of Section 5 in the language of Ramsey space (Corollary 4.1 & Theorem 5.1): The space  $\mathfrak{R}^{\vec{e}}(\mathbb{V}, \mathbb{F}, +_{\mathbb{V}}, +_{\mathbb{F}}, \times_{\mathbb{F}}, \cdot)$  is a Ramsey space if the field  $\mathbb{F}$  is a finite field; otherwise, it is a Ramsey space only if  $\vec{e}$  is constant with value 1.

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SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITI SAINS MALAYSIA, 11800 USM, MALAYSIA

*E-mail address:* teohzuyao@gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITI SAINS MALAYSIA, 11800 USM, MALAYSIA

*E-mail address:* dasmenteh@usm.my